The Misspecified and Semiparametric lower bounds and their application to inference problems with Complex Elliptically Symmetric (CES) distributed data

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Part II - Outline of the talk

- Why semiparametric models?
- CRB in parametric models with finite-dimensional nuisance parameters: classical approach.
- CRB in parametric models with finite-dimensional nuisance parameters: “Hilbert-space-based” approach.
- Extension to semiparametric models.
- Semiparametric interpretation of Real and Complex ES distributions.
- Examples.
Part II - Outline of the talk

Why semiparametric models?

CRB in parametric models with finite-dimensional nuisance parameters: classical approach

CRB in parametric models with finite-dimensional nuisance parameters: “Hilbert-space-based” approach

Extension to semiparametric models

Semiparametric interpretation of Real and Complex ES distributions

Examples
A parametric model $\mathcal{P}_\theta$ is defined as a set of pdfs that are parametrized by a finite-dimensional parameter vector $\theta$:

$$\mathcal{P}_\theta \triangleq \{ p_X(x_1, \ldots, x_M|\theta), \theta \in \Theta \subseteq \mathbb{R}^q \} .$$

The (lack of) knowledge about the phenomenon of interest is summarized in $\theta$ that needs to be estimated.

**Pros**: Parametric inference procedures are generally “simple” due to the finite dimensionality of $\theta$.

**Cons**: A parametric model could be too restrictive and a misspecification problem\(^1\) may occur [1,2,3,4,5,6].

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Non-parametric models

- A non-parametric model $\mathcal{P}_p$ is a collection of pdfs possibly satisfying some functional constraints (i.e. *symmetry*):

$$\mathcal{P}_p \triangleq \{ p_X(x_1, \ldots, x_M) \in \mathcal{K} \},$$

where $\mathcal{K}$ is some constrained set of pdfs.

- **Pros**: The risk of model misspecification is minimized.

- **Cons**: In non-parametric inference we have to face with infinite-dimensional estimation problem.

- **Cons**: Non-parametric inference may be a prohibitive task due to the large amount of required data.
A semiparametric model\(^2\) \(\mathcal{P}_{\theta,g}\) is a set of pdfs characterized by a finite-dimensional parameter \(\theta \in \Theta\) along with a function, i.e. an infinite-dimensional parameter, \(g \in \mathcal{L}\) [7]:

\[
\mathcal{P}_{\theta,g} \triangleq \{ p_X(x_1, \ldots, x_M | \theta, g), \theta \in \Theta \subseteq \mathbb{R}^q, g \in \mathcal{L} \}.
\]

Usually, \(\theta\) is the (finite-dimensional) parameter of interest while \(g\) can be considered as a nuisance parameter.

**Pros**: All parametric signal models involving an unknown noise distribution are semiparametric models.

**Cons**: Tools from functional analysis are needed.

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Examples: CES distributions

- A CES distributed random vector $\mathbf{x} \in \mathbb{C}^N$ admits a pdf [8]:

$$p_{\mathbf{x}}(\mathbf{x}; \mu, \Sigma) = c_{N,g} |\Sigma|^{-1} g((\mathbf{x} - \mu)^H \Sigma^{-1} (\mathbf{x} - \mu)),$$

- $c_{N,g}$ is a normalizing constant,
- $g \in \mathcal{G}$, $g : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$ is the density generator,
- $\mu \in \mathbb{C}^N$ is the mean value,
- $\Sigma \in \mathcal{M}_N$ is the (full rank) scatter matrix.

- The set of all CES pdfs is a semiparametric model of the form:

$$\mathcal{P}_{\mu, \Sigma, g} \overset{\triangle}{=} \left\{ p_{\mathbf{x}} | p_{\mathbf{x}}(\mathbf{x}|\mu, \Sigma, g), \mu \in \mathbb{C}^N, \Sigma \in \mathcal{M}_N, g \in \mathcal{G} \right\}.$$

- This semiparametric model is a particular instance of the more general set of *semiparametric group models* [9, Sec. 4.2].
Examples: Missing data

- Let \( \mathbf{z} \triangleq (\mathbf{x}^T, \mathbf{y}^T)^T \) be a complete dataset, where:
  - \( \mathbf{x} \) is the observed (available) dataset.
  - \( \mathbf{y} \) is the unobservable (missing) dataset.

- **Problem**: Estimate \( \mathbf{\theta} \in \Theta \) from the observed dataset \( \mathbf{x} \) when the pdf \( p_Y \) of the missing data \( \mathbf{y} \) is unknown.

- The pdf \( p_X \) of the observed dataset can be expressed as:
  
  \[
  p_X(\mathbf{x}|\mathbf{\theta}) = \int_Y p_{X,Y}(\mathbf{x}, \mathbf{y}|\mathbf{\theta})d\mathbf{y} = \int_Y p_{X|Y}(\mathbf{x}|\mathbf{y}, \mathbf{\theta})p_Y(\mathbf{y})d\mathbf{y}.
  \]

- The set of all the pdfs of the observed dataset \( \mathbf{x} \) is a **semiparametric mixture model** of the form [9, Sec. 4.5], [10]:
  
  \[
  \mathcal{P}_{\mathbf{\theta},p_Z} \triangleq \{ p_X|p_X(\mathbf{x}|\mathbf{\theta}, p_Y), \mathbf{\theta} \in \Theta, p_Y \in \mathcal{K} \}.
  \]
Examples: Non-linear regression

Let us consider the general non-linear regression model:

\[ x = f(z, \theta) + \epsilon, \]

- \( \theta \in \Theta \): parameter vector to be estimated,
- \( f \in \mathcal{F} \): possibly unknown non-linear function,
- \( z \): random vector with possibly unknown pdf \( p_Z \in \mathcal{K} \),
- \( \epsilon \): random noise with possibly unknown pdf \( p_\epsilon \in \mathcal{E} \)

The set of all pdfs for \( x \) is a semiparametric model of the form:

\[
\mathcal{P}_{\theta, f, p_Z, p_\epsilon} \triangleq \{ p_X(x|\theta, f, p_Z, p_\epsilon), \theta \in \Theta, f \in \mathcal{F}, p_Z \in \mathcal{K}, p_\epsilon \in \mathcal{E} \}.
\]

This model is a general form of a \textit{semiparametric regression model} [9, Sec. 4.3].
Examples: Autoregressive processes

- Consider the $AR(p)$ process:

$$x_n = \sum_{i=1}^{p} \theta_i x_{n-i} + w_n, \quad n \in (-\infty, \infty)$$

- $\theta \triangleq [\theta_1, \ldots, \theta_p]$: parameter vector to be estimated.
- $w_n$: i.i.d. innovations with unknown pdf $p_w \in \mathcal{W}$,

- Let $x \in \mathbb{R}^N$ a vector of $N$ observations from an $AR(p)$.

- The set of all possible pdfs for $x \in \mathbb{R}^N$ is a semiparametric model [11,12]:

$$\mathcal{P}_{\theta, p_w} \triangleq \{p_x | p_x(x|\theta, p_w), \theta \in \Theta, p_w \in \mathcal{W}\}.$$
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Examples
Let us consider the following *parametric model* involving a finite-dimensional vector of nuisance parameters:

\[ \mathcal{P}_{\theta, \eta} \triangleq \left\{ p_X(x|\theta, \eta), \theta \in \Theta \subseteq \mathbb{R}^q, \eta \in \Gamma \subseteq \mathbb{R}^d \right\}, \]

- \( \theta \in \Theta \): vector of the parameters of interest to be estimated,
- \( \eta \in \Gamma \): vector of the (unknown) nuisance parameters.

Denote with \( \theta_0 \) and \( \eta_0 \) the true value of \( \theta \in \Theta \) and \( \eta \in \Gamma \), respectively. Then \( p_0(x) \triangleq p_X(x|\theta_0, \eta_0) \).

**Score vectors** of the parametric model \( \mathcal{P}_{\theta, \eta} \) in \( \theta_0 \) and \( \eta_0 \):

\[ s_{\theta_0} \triangleq \nabla_{\theta} \ln p_X(x|\theta_0, \eta_0), \quad s_{\eta_0} \triangleq \nabla_{\eta} \ln p_X(x|\theta_0, \eta_0). \]
The Fisher Information Matrix (FIM)

The FIM for the parametric model $\mathcal{P}_{\theta, \eta}$ is given by:

$$I(\theta_0, \eta_0) \triangleq \begin{pmatrix} E_0 \left\{ s\theta_0 s_{\theta_0}^T \right\} & E_0 \left\{ s\theta_0 s_{\eta_0}^T \right\} \\ E_0 \left\{ s\eta_0 s_{\theta_0}^T \right\} & E_0 \left\{ s\eta_0 s_{\eta_0}^T \right\} \end{pmatrix} = \begin{pmatrix} I_{\theta_0\theta_0} & I_{\theta_0\eta_0} \\ I_{\theta_0\eta_0}^T & I_{\eta_0\eta_0} \end{pmatrix},$$

where $E_0 \{ h \} \triangleq \int h(x) p_0(x) \, dx$.

Let $\hat{\theta}(x)$ be an unbiased estimator of $\theta_0$: $E_0 \{ \hat{\theta}(x) \} = \theta_0$.

How can we derive the CRB on the estimation of $\theta_0$ in the presence of the unknown nuisance parameter vector $\eta_0$?
The Cramér-Rao inequality provides us with a lower bound on the error covariance matrix of $\hat{\theta}(x)$ when $\eta_0$ is unknown (see e.g. [13, Sec. 10.7]):

$$E_0 \left\{ (\hat{\theta}(x) - \theta_0)(\hat{\theta}(x) - \theta_0)^T \right\} \geq \text{CRB}(\theta_0|\eta_0).$$

**Classical approach**: $\text{CRB}(\theta_0|\eta_0)$ can be obtained from the FIM using the Matrix Inversion Lemma [14]:

$$\text{CRB}(\theta_0|\eta_0) \triangleq \left( I_{\theta_0\theta_0} - I_{\theta_0\eta_0} I_{\eta_0\eta_0}^{-1} I_{\eta_0\theta_0}^T \right)^{-1}.$$

It is possible to obtain this same result by using a geometrical, "Hilbert-space-based" approach [7].
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Examples
Hilbert spaces

Definition ([9, A.1, A.2],[15])

A Hilbert space $\mathcal{F}$ is a *normed vector space*

1. equipped with an *inner product* $\langle \cdot , \cdot \rangle$ and,
2. *complete* with respect to the norm $\| \cdot \| = \sqrt{\langle \cdot , \cdot \rangle}$.

- A normed (metric) space is complete when every Cauchy sequences in $\mathcal{F}$ converges to an element of $\mathcal{F}$.

- $f_1, f_2, \cdots$ is a Cauchy sequence if, for every $\varepsilon > 0$ there is a positive integer $N$ such that for all $i, j > N$, we have that:

$$\| f_i - f_j \| < \varepsilon.$$
The square-integrable functions

- Let \((\mathcal{X}, \mathcal{F}, \mu)\) be a measure space where \(\mathcal{X} \subseteq \mathbb{R}^N\), \(\mathcal{F}\) is the Borel \(\sigma\)-algebra on \(\mathcal{X}\) and \(\mu\) is a measure on \(\mathcal{F}\). \(^3\)

- Then, \(L_2(\mu)\) is the space of all the measurable functions s. t.

\[
L_2(\mu) = \left\{ f : \mathcal{X} \to \mathbb{R} \left| \int_{\mathcal{X}} |f(x)|^2 d\mu(x) < \infty \right. \right\}.
\]

- The \(L_2(\mu)\) space is an Hilbert space with the following inner product:

\[
\langle f_1, f_2 \rangle \triangleq \int_{\mathcal{X}} f_1(x)f_2(x)d\mu(x).
\]

- For the standard Lebesgue measure: \(d\mu(x) = dx\).

\(^3\)Some additional definitions are given in the backup slides.
The space of scalar zero-mean functions

- Let \(( \mathcal{X}, \mathcal{F}, P_X)\) be a probability space where \(\mathcal{X} \subseteq \mathbb{R}^N\) is the sample space, \(\mathcal{F}\) is the Borel \(\sigma\)-algebra on \(\mathcal{X}\) and \(P_X\) is a probability measure. \(^4\)

- Let \(\mathcal{H}\) be the Hilbert space defined as [10, Ch. 2]:

\[
\mathcal{H} = \{ h : \mathcal{X} \to \mathbb{R} \mid E_X\{ h \} = 0, E_X\{ |h|^2 \} < \infty \}.
\]

- The expectation operator \(E_X\{ \cdot \}\) is

\[
E_X\{ h \} \triangleq \int_{\mathcal{X}} h(x) dP_X(x) = \int_{\mathcal{X}} h(x) p_X(x) dx,
\]

where \(p_X\) is the probability density function (pdf).

- The inner product in \(\mathcal{H}\) is: \(\langle h_1, h_2 \rangle \triangleq E_X\{ h_1 h_2 \}\).

\(^4\)Some additional definitions are given in the backup slides.
The projection theorem (1/2)

Theorem

Let $\mathcal{U}$ be a closed subspace of an Hilbert space $\mathcal{F}$ and take $f \in \mathcal{F}$. We call

$$d(f, \mathcal{U}) \triangleq \inf_{u \in \mathcal{U}} \|f - u\|, \quad f \in \mathcal{F},$$

the distance of $f$ to $\mathcal{U}$. Then there exists a unique element $\tilde{u} \in \mathcal{U}$ for which

$$\|f - \tilde{u}\| = d(f, \mathcal{U}).$$

\[\[\]
\]
The projection theorem (2/2)

- $f$ can be uniquely written as:

$$f = \tilde{u} + (f - \tilde{u}),$$

where $\tilde{u} \triangleq \Pi(f|\mathcal{U}) \in \mathcal{U}$ and $f - \tilde{u} \in \mathcal{U}^\perp$.

- $\tilde{u}$ is uniquely determined by the orthogonality constraint:

$$\langle f - \tilde{u}, u \rangle = \langle f - \Pi(f|\mathcal{U}), u \rangle = 0, \quad \forall u \in \mathcal{U}.$$
The linear span

- A \textit{\textit{q-replicating}} Hilbert space $\mathcal{F}^q$ is obtained by the Cartesian product of the $q$ copies of $\mathcal{F}$ as $\mathcal{F}^q \triangleq \mathcal{F} \times \cdots \times \mathcal{F}$, then:

$$\mathcal{F}^q \ni \mathbf{f} = (f_1, f_2, \cdots, f_q)^T, \quad f_i \in \mathcal{F}.$$ 

- The inner product of $\mathcal{F}^q$ is induced by the one in $\mathcal{F}$:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{i=1}^{q} \langle f_i, g_i \rangle.$$ 

- \textbf{Linear span}: Let $\mathbf{u} = (u_1, \cdots, u_k)^T$ be a column vector of $k$ elements of $\mathcal{F}$. The \textit{linear span} of the vector $\mathbf{u}$, defined as:

$$\mathcal{V} \triangleq \{ \mathbf{v} | \mathbf{v} = \mathbf{A} \mathbf{u}, \mathbf{A} \text{ is any matrix in } \mathbb{R}^{q \times k} \},$$

is a \textit{finite-dimensional} subspace of $\mathcal{F}^q$. 
Projection onto a finite-dimensional subspace

\[ \mathcal{V} \triangleq \{ \mathbf{v} | \mathbf{v} = \mathbf{A} \mathbf{u}, \mathbf{A} \text{ is any matrix in } \mathbb{R}^{q \times k} \} . \]

- If \( u_1, \ldots, u_k \) are linearly independent in \( \mathcal{F} \), \( \dim(\mathcal{V}) = kq \). \(^5\)

- The projection of a generic element \( \mathbf{f} \in \mathcal{F}^q \) onto the subspace \( \mathcal{V} \) is given by [9, A.2], [10, Sec. 2.4]:

\[
\Pi(\mathbf{f} | \mathcal{V}) = \left( \langle \mathbf{f}, \mathbf{u}^T \rangle \langle \mathbf{u}, \mathbf{u}^T \rangle \right)^{-1} \mathbf{u},
\]

where

\[
\left[ \langle \mathbf{f}, \mathbf{u}^T \rangle \right]_{i,j} \triangleq \langle f_i, u_j \rangle, \quad i = 1, \ldots, q,
\]

\[
\left[ \langle \mathbf{u}, \mathbf{u}^T \rangle \right]_{i,j} \triangleq \langle u_i, u_j \rangle, \quad i, j = 1, \ldots, k.
\]

\(^5\) The proof of this result is in the backup slides (see also [10, Sec. 2.4]).
The vector-valued zero-mean functions

- Let \((\mathcal{X}, \mathcal{F}, P_X)\) be a probability space.

- Let \(\mathcal{H}^q\) be the \(q\)-replicating Hilbert space [10, Ch. 2]:

  \[
  \mathcal{H}^q = \mathcal{H} \times \cdots \times \mathcal{H}
  = \left\{ h : \mathcal{X} \to \mathbb{R}^q \left| E_X\{h\} = 0, E_X\{h^T h\} < \infty \right. \right\},
  \]

- The induced inner product is:

  \[
  \langle h_1, h_2 \rangle \triangleq E_X\{h_1^T h_2\}.
  \]

- The covariance matrix of \(h \in \mathcal{H}^q\) is:

  \[
  C_X(h) \triangleq E_X\{hh^T\}.
  \]
Let \( u = (u_1, \cdots, u_k)^T \) be a column vector of \( k \) arbitrary elements of \( \mathcal{H} \) and let \( \mathcal{V} \) be its linear span.

The orthogonal projection of an arbitrary element \( h \in \mathcal{H}^q \) onto \( \mathcal{V} \) is unique and it is given by [9, A.2], [10, Sec. 2.4]:

\[
\Pi(h|\mathcal{V}) = E_X \{hu^T\} E_X \{uu^T\}^{-1}u = E_X \{hu^T\} C_X(u)^{-1}u.
\]

Linear Minimum Mean Square Error (LMMSE) estimator:

1. \( \text{MSE} \triangleq \|h - Au\|^2 \) is minimized by \( \Pi(h|\mathcal{V}) \), then
   \[
   \hat{h}_{LMMSE} = E_X \{hu^T\} C_X(u)^{-1}u.
   \]

2. The “orthogonality principle” is nothing but the Projection Theorem.
Score vectors as elements of $\mathcal{H}^r$ (1/2)

Let us go back to the parametric model:

$$\mathcal{P}_{\theta, \eta} \triangleq \left\{ p_X(x|\theta, \eta), \theta \in \Theta \subseteq \mathbb{R}^q, \eta \in \Gamma \subseteq \mathbb{R}^d \right\},$$

- $\theta \in \Theta$ is the vector of the parameters of interest,
- $\eta \in \Gamma$ is the vector of the (unknown) nuisance parameters,
- $\gamma \triangleq (\theta^T, \eta^T)^T \in \mathbb{R}^r$, $r = q + d$.
- $p_0(x) \triangleq p_X(x|\theta_0, \eta_0)$ is the "true" pdf.

The score vector for the true parameter vector $\gamma_0$ is:

$$s_{\gamma_0} \triangleq \nabla_{\gamma} \ln p_X(x|\gamma_0) = \begin{pmatrix} \nabla_{\theta} \ln p_X(x|\theta_0, \eta_0) \\ \nabla_{\eta} \ln p_X(x|\theta_0, \eta_0) \end{pmatrix} \triangleq \begin{pmatrix} s_{\theta_0} \\ s_{\eta_0} \end{pmatrix}$$

- $s_{\theta_0}$ is $q \times 1$ the score vector of the parameters of interest,
- $s_{\eta_0}$ is $d \times 1$ the nuisance score vector.
Score vectors as elements of $\mathcal{H}^r$ (2/2)

Under standard regularity conditions [16]:

$$E_0 \{s_{\gamma_0}\} = \int_{\mathcal{X}} \nabla_{\gamma} \ln p_X(x|\gamma_0) dP_0(x)$$

$$= \int_{\mathcal{X}} \frac{\nabla_{\gamma} p_X(x|\gamma_0)}{p_0(x)} p_0(x) dx = \nabla_{\gamma} \int_{\mathcal{X}} p_X(x|\gamma_0) dx = 0,$$

and $E_0 \{s_{\gamma_0}^T s_{\gamma_0}\} < \infty$.

Then, by definition$^6$ of $\mathcal{H}^r$:

$$\mathcal{H}^r \ni s_{\gamma_0} = \begin{pmatrix} s_{\theta_0} \\ s_{\eta_0} \end{pmatrix} \Rightarrow s_{\theta_0} \in \mathcal{H}^q, \quad s_{\eta_0} \in \mathcal{H}^d.$$
The efficient score vector

- The *nuisance tangent space* $\mathcal{T}_{\eta_0}$ is defined as the linear span of $s_{\eta_0}$ in $\mathcal{H}^q$ [10, Ch. 3]:

$$\mathcal{T}_{\eta_0} \triangleq \{ t \mid t = A s_{\eta_0}, A \text{ is any matrix in } \mathbb{R}^{q \times d} \} \subset \mathcal{H}^q.$$

- Let us define the **efficient score vector** as [9, Ch. 2]:

$$\bar{s}_0 \triangleq s_{\theta_0} - \Pi(s_{\theta_0} \mid \mathcal{T}_{\eta_0})$$

$$= s_{\theta_0} - E\{ s_{\theta_0} s_{\eta_0}^T \}_{\eta_0} l^{-1} s_{\eta_0}.$$

---

7 The geometrical intuition behind this terminology is given in the backup slides.
Evaluation of the CRB using $\bar{s}_0$

- $\bar{s}_0$ is the residual of $s_{\theta_0}$ after projecting it onto the nuisance tangent space $T_{\eta_0}$.

- Let us define the efficient FIM as:

$$\bar{I}(\theta_0|\eta_0) \triangleq E_0 \left\{ \bar{s}_0\bar{s}_0^T \right\}.$$ 

- Through direct calculation, we get:

$$\bar{I}(\theta_0|\eta_0) = I_{\theta_0\theta_0} - I_{\theta_0\eta_0} I_{\eta_0\eta_0}^{-1} I_{\eta_0\theta_0}^T.$$ 

- The inverse of $\bar{I}(\theta_0|\eta_0)$ is exactly the CRB($\theta_0|\eta_0$) previously derived by means of the Matrix Inversion Lemma:

$$\left[ E \left\{ \bar{s}_0\bar{s}_0^T \right\} \right]^{-1} \triangleq \left[ \bar{I}(\theta_0|\eta_0) \right]^{-1} = \text{CRB}(\theta_0|\eta_0).$$
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The three basic ingredients

- In summary, to derive the \( \text{CRB}(\theta_0|\eta_0) \), we only need:

  1. The Hilbert space \( \mathcal{H}^q \),
  2. The nuisance tangent space \( \mathcal{T}_{\eta_0} \subseteq \mathcal{H}^q \) of the parametric model \( P_{\theta,\eta} \) at \( \eta_0 \),
  3. The projection operator onto \( \mathcal{T}_{\eta_0} \): \( \Pi(s_{\theta_0}|\mathcal{T}_{\eta_0}) \).

- **Important fact**: None of them require the finite dimensionality of the nuisance parameters [7].

- This alternative way to calculate the CRB can be extended to semiparametric models.

- To make this extension possible, we have to introduce the concept of parametric submodel.
Let us recall the semiparametric model:

$$\mathcal{P}_{\theta,g} \triangleq \{p_X(x|\theta, g), \theta \in \Theta \subseteq \mathbb{R}^q, g \in \mathcal{L}\}.$$ 

The i-th parametric submodel\(^8\) of \(\mathcal{P}_{\theta,g}\) is defined as [10, Sec. 4.2], [9, Sec. 3.1], [17,18,11], :

$$\mathcal{P}_{\theta,\nu_i} = \{p_X(x|\theta, \nu_i(x, \eta)), \theta \in \Theta, \eta \in \Gamma_i\},$$

where:

$$\nu_i : \Gamma_i \rightarrow \mathcal{L}$$

$$\eta \mapsto \nu_i(\cdot, \eta),$$

The function \(\nu_i \in \mathcal{L}\) is a known function parametrized by a vector of unknown parameters.

\(^8\) An explicit example of parametric submodel is given in the backup slides.
Parametric submodels (2/3)

- Denote the “true semiparametric vector” and the related true pdf as \((\theta_0^T, g_0)^T\) and \(p_0(x) \triangleq p_X(x|\theta_0, g_0)\), respectively.

- For every \(i \in \mathcal{I}\), the \(i\)-th parametric submodel:

\[
\mathcal{P}_{\theta, \nu_i} = \{ p_X(x|\theta, \nu_i(x, \eta), \theta \in \Theta, \eta \in \Gamma_i \},
\]

has to satisfy the following three conditions [10, Sec. 4.2]:

C0) \(\nu_i : \Gamma_i \to \mathcal{L}\) is a smooth parametric map,

C1) \(\mathcal{P}_{\theta, \nu_i} \subseteq \mathcal{P}_{\theta, g}\),

C2) \(p_0(x) \in \mathcal{P}_{\theta, \nu_i}\), i.e. there exists a vector \((\theta_0^T, \eta_0^T)^T\) such that \(p_X(x|\theta_0, \nu_i(x, \eta_0)) = p_X(x|\theta_0, g_0) \triangleq p_0(x)\).
The generalization to the semiparametric framework can be done in two steps:

1. Exploit the obtained results in the set of (artificial) parametric submodels \( \{ \mathcal{P}_{\theta, \nu_i} \}_{i \in \mathcal{I}} \),

2. “Take the limit” to generalize them in the infinite-dimensional semiparametric framework.
Semiparametric nuisance tangent space (1/2)

- For every parametric submodel:

\[ \mathcal{P}_{\theta, \nu_i} = \{ p_X(x|\theta, \nu_i(x, \eta)), \theta \in \Theta, \eta \in \Gamma_i \} , \]

we have a relevant nuisance tangent space:

\[ \mathcal{T}_{\eta_0,i} \equiv \{ t_i | t_i = A_i s_{\eta_0,i} : A_i \text{ is any matrix in } \mathbb{R}^{q \times d_i} \} , \]

where \( s_{\eta_0,i} \equiv \nabla_\eta \ln p_X(x|\theta_0, \nu_i(x, \eta_0)) \).

- The **semiparametric nuisance tangent space** is defined as:\(^9\)

\[ \mathcal{T}_{g_0} \equiv \bigcup_{\{ \mathcal{P}_{\theta, \nu_i} \}_{i \in I}} \mathcal{T}_{\eta_0,i} \subseteq \mathcal{H}^q . \]

---

\(^9\) The closure \( \bar{A} \) of a set \( A \) is defined as the smallest closed set that contains \( A \), or equivalently, as the set of all elements in \( A \) together with all the limit points of \( A \).
Recall that the Hilbert space $\mathcal{H}^q$ is a complete normed space with norm:

$$\|h_1 - h_2\| = \sqrt{E_0\{(h_1 - h_2)^T(h_1 - h_2)\}}, \quad \forall h_1, h_2 \in \mathcal{H}^q.$$ 

The semiparametric nuisance tangent space $\mathcal{T}_{g_0} \subseteq \mathcal{H}^q$ can be expressed as [10, Sec. 4.4],[19],[18]:

$$\mathcal{T}_{g_0} \triangleq \{h \in \mathcal{H}^q | \forall \varepsilon > 0, \exists i \in I : \|h - A_i s_{\eta_0,i}\| < \varepsilon\}$$

Unlike $\mathcal{T}_{\eta_0,i}$ that has finite dimension, $\mathcal{T}_{g_0}$ is in general an infinite-dimensional subspace of $\mathcal{H}^q$.

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10 A more explicit definition of the nuisance tangent space requires the notion of Hellinger differentiability [19],[9, Sec. 3.2]. See also the backup slides.
The projection operator $\Pi(\cdot | \mathcal{T}_{g_0})$

- The existence and the uniqueness of the projection operator $\Pi(\cdot | \mathcal{T}_{g_0})$ is guaranteed by the Projection Theorem.

- The semiparametric efficient score vector for the estimation of $\theta_0 \in \Theta$ in the presence of the nuisance function $g_0 \in \mathcal{L}$ is [9, Sec. 3.3]:

$$\bar{s}_0 \triangleq s_{\theta_0} - \Pi(s_{\theta_0} | \mathcal{T}_{g_0}).$$
Theorem ([9, Sec. 3.4], [19], [10, Theo. 4.2], [18]):
A lower bound on the MSE of “any” \(^{11}\) robust estimator of \(\theta_0\) in the presence of the nuisance function \(g_0 \in \mathcal{L}\) is given by:

\[
\text{SCRB}(\theta_0|g_0) = [\bar{I}(\theta_0|g_0)]^{-1},
\]

where \(\bar{I}(\theta_0|g_0) \triangleq E_0\{\bar{s}_0\bar{s}_0^T\}\) is the \textit{semiparametric FIM} (SFIM) and:

\[
\bar{s}_0 \triangleq s_{\theta_0} - \Pi(s_{\theta_0}|\mathcal{T}_{g_0}).
\]


\(^{11}\) The class of estimators to which the SCRB applies is discussed ahead.
The expression of $\text{SCRB}(\theta_0|g_0)$ is formally equivalent to $\text{CRB}(\theta_0|\eta_0)$ derived for finite-dimensional nuisance vectors.

The Hilbert-space-based approach allows to handle both finite and infinite-dimensional nuisance parameters.

The $\text{SCRB}(\theta_0|g_0)$ is higher than any $\text{CRB}(\theta_0|\eta_{0,i})$ derived in the $i$-th parametric submodel.

A semiparametric model contains less information on $\theta_0$ than any of its possible parametric submodel.
A bound for any robust estimator

- The SCRB is a lower bound for the MSE of any Regular and Asymptotically Linear (RAL) estimator [9, Sec. 2.2 and Ch. 7], [10, Ch.3], [20, Ch. 4] [21,18,22,23].

- All the robust $M$-, $S$-, $L$- estimators belong to this class [24]:

- It can be shown that every RAL estimator is:
  1. Consistent: $\hat{\theta}(x_1, \ldots, x_M) \triangleq \hat{\theta}_M \xrightarrow{M \to \infty} \theta_0$, 
  2. Asymptotically normal: $\sqrt{M}(\hat{\theta}_M - \theta_0) \xrightarrow{M \to \infty} \mathcal{N}(0, \Xi(\theta_0, g_0))$.

- Consequently, the following inequality holds [9, Ch. 2 and 3]:
  $$\Xi(\theta_0, g_0) \geq \text{SCRB}(\theta_0 | g_0).$$

- Note that efficient estimators may not exist [25].
Evaluation of $\Pi(\cdot | \mathcal{T}_{g_0})$

- The crucial step to evaluate $\text{SCRB}(\theta_0 | g_0)$ is in determining the semiparametric efficient score vector:

  $$\tilde{s}_0 \triangleq s_{\theta_0} - \Pi(s_{\theta_0} | \mathcal{T}_{g_0}).$$

- To this end, we need to:
  1. Calculate $s_{\theta_0} = \nabla_{\theta} \ln p_X(x | \theta_0, g_0)$ (easy task),
  2. Evaluate the projection $\Pi(s_{\theta_0} | \mathcal{T}_{g_0})$ (difficult task).

- Two possible approaches:
  1. Least Favourable Submodel (if it exists) \(^{12}\),
  2. Projection as a conditional expectation.

\(^{12}\)Some additional details are given in the backup slides.
Projection and conditional expectation (1/3)

- We defined $\mathcal{H}^q$ as the Hilbert space of the $q$-dimensional zero-mean function on the probability space $(\mathcal{X}, \mathcal{F}, P_X)$:

$$h \equiv h(x), \quad x \in \mathcal{X} \subseteq \mathbb{R}^N.$$  

- Let $f : \mathbb{R}^N \to \mathbb{R}$ be a measurable function. We define a statistic $V$ of the random vector $x$ as:

$$V =_d f(x), \quad \forall x \in \mathcal{X}.$$  

- Let $\mathcal{G}(V) \subseteq \mathcal{F}$ be the sub-$\sigma$ algebra generated by $V$.  

- The set of the $q$-dim zero-mean functions on $(\mathcal{X}, \mathcal{G}(V), P_X)$ is a closed linear subspace, say $\mathcal{V}$, of $\mathcal{H}^q$ [26, Theo. 23.2].

\(^{13}\) Additional details are given in the backup slides.
Let \( r \in \mathcal{H}^q \) be a zero-mean function of \( x \in \mathcal{X} \) through the function \( f \), i.e.: 14

\[
r \equiv r(f(x)) =_{d} r(V) \in \mathcal{V} \subseteq \mathcal{H}^q.
\]

Consequently, \( r \in \mathcal{H}^q \) can be considered as a \( q \)-dimensional function defined on \((\mathcal{X}, \mathcal{G}(\mathcal{V}), \mathcal{P}_X)\) with \( \mathcal{G}(\mathcal{V}) \subseteq \mathfrak{F} \).

![Diagram showing the projection and conditional expectation](image)

\(^{14}\) The symbol \( "=_{d}" \) means "has the same distribution as".
The conditional expectation $E\{h|V\}$ is the unique element in $\mathcal{V}$, such that [26, Def. 23.3, Theo. 23.3]$^{15}$:

$$\langle h - E\{h|V\}, r \rangle \triangleq E \left\{ (h - E\{h|V\})^T r \right\} = 0, \quad \forall r \in \mathcal{V}.$$ 

Given the Projection Theorem, the previous definition implies:

$$\Pi(\cdot|\mathcal{V}) = E\{\cdot|\mathcal{V}\}.$$ 

---

$^{15}$This definition is consistent with the classical one [26, Ch. 32]. See the proof in the backup slides.
Part II - Outline of the talk

Why semiparametric models?

CRB in parametric models with finite-dimensional nuisance parameters: classical approach

CRB in parametric models with finite-dimensional nuisance parameters: “Hilbert-space-based” approach

Extension to semiparametric models

Semiparametric interpretation of Real and Complex ES distributions

Examples
Spherically Symmetric (SS) distributions

- Let $z \in \mathbb{R}^N$ be a real-valued random vector.

- Let $\mathcal{O}$ be the set of all unitary transformations:

  $$\mathcal{O} \ni O : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

  $$z \mapsto O(z) = Oz,$$

  for any unitary matrix $O$, i.e. $O^T O = O O^T = I$.

- Then, $z$ is said to be SS-distributed if its distribution is invariant to any unitary transformations $O \in \mathcal{O}$, i.e.

  $$z =_d Oz.$$

- We indicate with $S$ the class of all SS-distributions.
Property P1  

The SS-distributed random vector $z \sim SS(g)$ has a pdf:

$$p_Z(z) = 2^{-N/2} g \left( \|z\|^2 \right),$$

where $G \ni g$, is a function, called density generator and

$$G = \left\{ g : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+ \left| \int_0^\infty t^{N/2-1} g(t) dt < \infty \right. \right\}.$$

The set of all SS pdfs can be described as:

$$S = \left\{ p_Z | p_Z(z) = 2^{-N/2} g \left( \|z\|^2 \right), \forall g \in G \right\}.$$
Property P2

- Let $s_N \triangleq 2\pi^{N/2}/\Gamma(N/2)$ be the surface area of the unit sphere $\mathbb{R}S^N$ in $\mathbb{R}^N$.

- The pdf of $Q =_d ||z||^2$, called 2nd-order modular variate, is:

$$p_Q(q) = s_N 2^{-N/2-1} q^{N/2-1} g(q).$$

- The pdf of $R \triangleq \sqrt{Q} =_d ||z||$, called modular variate, is:

$$p_R(r) = s_N 2^{-N/2} r^{N-1} g(r^2).$$
Property P3: *Stochastic Representation Theorem*

- Let \( \mathbf{u} \sim \mathcal{U}(\mathbb{R}S^N) \) be a random vector uniformly distributed on \( \mathbb{R}S^N \), i.e. \( \|\mathbf{u}\| = 1 \).

- If \( \mathbf{z} \in \mathbb{R}^N \) is SS-distributed, i.e. \( \mathbf{z} \sim SS(g) \), then:
  \[
  \mathbf{z} = d \sqrt{\mathbf{Q}} \mathbf{u} = d \mathbf{R} \mathbf{u},
  \]

- Moreover, \( \mathbf{Q} \) and \( \mathbf{u} \) (or \( \mathbf{R} \) and \( \mathbf{u} \)) are independent.

- P2 and P3 imply that, not knowing the density generator \( g \) has an impact only on the pdf of the r.v. \( \mathbf{R} \) (or \( \mathbf{Q} \)).
Property P4: *Invariant statistic*

- By definition of SS distributions, $\| \cdot \|$ is an *invariant statistic* since [30, Ch. 6]
  \[
  \|z\| =_d \|Oz\|,
  \]
  for every unitary matrix $O \in \mathcal{O}$.

- Moreover, given two SS-distributed r.v. $z_1$ and $z_2$, we have:
  \[
  \|z_1\| =_d \|z_2\| \Rightarrow z_1 =_d Oz_2, \quad \forall O \in \mathcal{O}.
  \]

- Then, the modular variate $R =_d \|z\|$ is a *maximal invariant statistic* for the set of the SS-distributed random vectors.
Tangent space and invariance

Let $\mathcal{A}$ be a group of transformations from $\mathbb{R}^N$ into itself:

$$\mathcal{A} \ni \alpha : \mathbb{R}^N \to \mathbb{R}^N$$

$$z \mapsto \alpha(z),$$

Suppose that $\mathcal{P}$ is a set of pdfs which are invariant with respect to $\mathcal{A}$, i.e.:

$$\mathcal{P} = \left\{ p_Z \, | \, p_Z(\alpha(z)) = p_Z(z); \forall \alpha \in \mathcal{A}, \forall z \in \mathbb{R}^N \right\}.$$

Then, the tangent space $\mathcal{T}$ of $\mathcal{P}$ is given by [9, App. 3]:

$$\mathcal{T} = \left\{ h \in \mathcal{H} \, | \, h(\alpha(z)) = h(z), \forall \alpha \in \mathcal{A}, \forall z \in \mathbb{R}^N \right\}.$$

---

\[\text{Remember that } \mathcal{H} = \left\{ h : \mathcal{X} \to \mathbb{R} \, | \, E_X \{h\} = 0, E_X \{|h|^2\} < \infty \right\}.\]
Projection and invariance

If there exists an invariant statistic $D$ for $z \sim p_Z$ s.t.:

$$D = d \quad D(\alpha(z)), \quad \forall \alpha \in \mathcal{A},$$

then the projection operator on $\mathcal{T}$ can be calculated as [9, App. 3]:

$$\Pi(\cdot|\mathcal{T}) = E\{\cdot|D\}.$$

Example: SS distributions

- The tangent space $\mathcal{T}_S$ is given by:

$$\mathcal{T}_S = \left\{ h \in \mathcal{H} \mid h(\|z\|) = h(z), \forall z \in \mathbb{R}^N \right\},$$

- $\Pi(\cdot|\mathcal{T}_S) = E\{\cdot|\mathcal{R}\}$ where $\mathcal{R} = d \quad \|z\|$ is the modular variate.
Let $\mathcal{A}$ be a group of *parametric* transformations from $\mathbb{R}^N$ into itself:

$$\mathcal{A} = \{ \alpha| \alpha(\cdot; \theta) \triangleq \alpha_\theta(\cdot); \theta \in \Theta \subseteq \mathbb{R}^q \}.$$  

- $\alpha^{-1}_\theta(\cdot)$ defines the inverse of $\alpha_\theta(\cdot)$,
- $(\alpha_{\theta_2} \circ \alpha_{\theta_1})(\cdot) \triangleq \alpha_{\theta_2}(\alpha_{\theta_1}(\cdot))$ denotes the composition,
- $\theta_e$ indicates the parameter vector that characterizes the identity transformation $\alpha_{\theta_e}$, s.t. $\alpha_{\theta_e}(\cdot) = \cdot$.

**Example:** Let us define $\theta \triangleq [\mu, \sigma]^T$, then:

$$\alpha_\theta(z) \triangleq \mu + \sigma z,$$

$$\alpha^{-1}_\theta(z) = (z - \mu)/\sigma, \quad \theta_e \triangleq [0, 1]^T.$$
Let $z \in \mathbb{R}^N$ be a random vector s.t. $z \sim p_Z(z)$.

The *parametric group model*, generated by the action of $\mathcal{A}$ on $z$ can be expressed as:

$$\mathcal{P}_\theta = \{ p_X | p_X(x|\theta) = |J(\alpha^{-1}_\theta(x))|p_Z(\alpha^{-1}_\theta(x)); \theta \in \Theta \},$$

where:

- $[J(\alpha^{-1}_\theta(x))]_{i,j} \triangleq \partial[\alpha^{-1}(x; \theta)]_i/\partial \theta_j$ is the Jacobian matrix of the inverse transformation $\alpha^{-1}_\theta$,
- $|\cdot|$ defines the (absolute value of the) determinant of the Jacobian matrix.
If $p_Z$ is allowed to vary in a function set $L$, we get a semiparametric group model:

$$P_{\theta, p_Z} = \{ p_X | p_X(x|\theta, p_Z) = |J(\alpha_{\theta}^{-1})(x)|p_Z(\alpha_{\theta}^{-1}(x)); \theta \in \Theta, p_Z \in L \}.$$ 

The calculation of the projection operator can be greatly simplified!

1. Evaluate the projection on the semiparametric nuisance tangent space at the identity $\alpha_{\theta_e}$.
2. “Translate” the projection in any other $\theta$ of the parameter space $\Theta$.  

Semiparametric group models (1/2)
Semiparametric group models (2/2)

- $\mathcal{T}_{pZ,0}(\theta_e) \subseteq \mathcal{H}^q$: Semiparametric nuisance tangent space at the identity $\theta_e$.

- $\mathcal{T}_{pZ,0}(\theta) \subseteq \mathcal{H}^q$: Semiparametric nuisance tangent space at a generic $\theta \in \Theta$.

The projection operator on $\mathcal{T}_{pZ,0}(\theta)$ can be obtained as [9, Sec. 4.2, Lemma 3]:

$$\Pi(\cdot | \mathcal{T}_{pZ,0}(\theta)) = \Pi(\cdot \circ \alpha_\theta | \mathcal{T}_{pZ,0}(\theta_e)) \circ \alpha_\theta^{-1}, \quad \forall \theta \in \Theta.$$
Let us define the parameter space $\Theta \subseteq \mathbb{R}^q$ as:

$$\Theta = \left\{ \theta \in \mathbb{R}^q | \theta = [\mu^T, \text{vecs}(\Sigma)^T]^T ; \mu \in \mathbb{R}^N, \Sigma \in \mathcal{M}_N \right\}.$$

We can define the group of parametric transformations $\mathcal{A}$ as:

$$\mathcal{A} \ni \alpha_\theta : \mathbb{R}^N \to \mathbb{R}^N, \forall \theta \in \Theta$$

$$z \mapsto \alpha_\theta(z) = \mu + \Sigma^{1/2}z.$$

The identity $\alpha_{\theta_e}$ is parametrized by $\theta_e = [0^T, \text{vecs}(I)^T]^T$.

The inverse is simply given by:

$$\alpha_\theta^{-1}(\cdot) = \Sigma^{-1/2}(\cdot - \mu).$$
From SS to RES distributions (2/2)

- A random vector \( x \in \mathbb{R}^N \) is said to be RES-distributed if it can be expressed as:

\[
x = \alpha_\theta(z) = \mu + \Sigma^{1/2}z = d \mu + \mathcal{R}\Sigma^{1/2}u,
\]

- \( z \sim SS(g) \) is an SS-distributed random vector,

- \( u \sim \mathcal{U}(\mathbb{R}^S) \) and \( \mathcal{R} = \sqrt{Q} \) is the modular variate, s.t.:

\[
Q = d \| z \|^2 = \| \alpha_\theta^{-1}(x) \|^2 = (x - \mu)^T \Sigma^{-1}(x - \mu).
\]

- RES distributions represent a semiparametric group model:

\[
P_{\theta, g} = \left\{ px \mid px(x|\theta, g) = 2^{-N/2} |\Sigma|^{-1/2} g(\| \alpha_\theta^{-1}(x) \|^2); \theta \in \Theta, g \in G \right\},
\]
Part II - Outline of the talk

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Examples
Evaluation of the SCRB for the RES class

\[
p(x|\theta_0, g_0) = 2^{-N/2}|\Sigma_0|^{-1/2}g((x - \mu_0)^T \Sigma_0^{-1}(x - \mu_0)),
\]

\[
\theta_0 = [\mu_0^T, \text{vecs}(\Sigma_0)^T]^T.
\]

**Problem:** Find the (Constrained) SCRB on the estimation of the mean vector \( \mu_0 \) and of the scatter matrix \( \Sigma_0 \) when the density generator \( g_0 \) is unknown.

To avoid the ambiguity between \( \Sigma_0 \) and \( g_0 \), we put a constraint on the scatter matrix:

\[
c(\Sigma_0) = 0.
\]

All the details can be found in [29].
Evaluation of the SCRB for the RES class

Step A: Evaluation of the score vector $s_{\theta_0}$

By definition:

$$s_{\theta_0} = \nabla_{\theta} \ln p_X(x|\theta_0, g_0) = \begin{pmatrix} s_{\mu_0} \\ s_{\text{vecs}(\Sigma_0)} \end{pmatrix}$$

Through direct calculation, we get:

$$s_{\mu_0} = d - 2\sqrt{Q}\psi_0(Q)\Sigma_0^{-1/2}u,$$

$$s_{\text{vecs}(\Sigma_0)} = d - D_N^T \left( 2^{-1} \text{vec}(\Sigma_0^{-1}) + Q\psi_0(Q)\Sigma_0^{-1/2} \otimes \Sigma_0^{-1/2} \text{vec}(uu^T) \right),$$

$\psi_0(t) \triangleq d \ln g_0(t)/dt,$

 Duplication matrix: $D_N\text{vecs}(A) = \text{vec}(A), \ \forall A$ symmetric.
Step B: Evaluation of the projection operator $\Pi(s_{\theta_0}|T_{g_0})$

- Due to the group structure underlying the RES class, $T_{g_0}$ evaluated at the group identity $\theta_e$ is given by:

$$T_{g_0}(\theta_e) = \{l|l = ha; h \in T_S, a \in \mathbb{R}^q\};$$

where $T_S$ is the tangent space of the SS distributions:

$$T_S = \left\{ h \in \mathcal{H} | h(||x||) = h(x), \forall x \in \mathbb{R}^N \right\},$$

- Using the property of the semiparametric group model:

$$\Pi(s_{\theta_0}|T_{g_0}(\theta_0)) = \Pi(s_{\theta_0} \circ \alpha_{\theta_0}|T_{g_0}(\theta_e)) \circ \alpha_{\theta_0}^{-1}$$

$$= E \{s_{\theta_0} \circ \alpha_{\theta_0}|\mathcal{R}\} \circ \alpha_{\theta_0}^{-1}. $$
Evaluation of the SCRB for the RES class

Through direct calculation (see [29] for the details):

\[
\prod(s_{\theta_0} \mid T_{g_0}) = \begin{pmatrix}
\prod(s_{\mu_0} \mid T_{g_0}) \\
\prod(s_{\text{vecs}(\Sigma_0)} \mid T_{g_0})
\end{pmatrix}
= d \begin{pmatrix}
0 \\
-D N^T \left( \frac{1}{2} + \frac{1}{N} Q \psi_0(Q) \right) \text{vec}(\Sigma_0^{-1})
\end{pmatrix}.
\]

The score function \(s_{\mu_0}\) of the mean value is orthogonal to the nuisance tangent space \(T_{g_0}\),

Not knowing the true \(g_0\) does not have any impact in the (asymptotic) estimation performance of \(\mu_0\) [21].
Evaluation of the SCRB for the RES class

Step C: Evaluation of the semiparametric FIM \( \bar{I}(\theta_0, g_0) \)

- The efficient score vector \( \bar{s}_0 \) can then be expressed as:

\[
\bar{s}_0 = s_{\theta_0} - \Pi(s_{\theta_0}(x)|T_{g_0})
\]

\[
= d \begin{pmatrix}
-2\sqrt{Q_0}(Q)\Sigma_0^{-1/2}u \\
-D_N^TQ_0(Q)\left(\Sigma_0^{-1/2} \otimes \Sigma_0^{-1/2}\vec{\text{vec}(uu^T)} - \frac{\vec{\text{vec}(\Sigma_0^{-1})}}{N}\right)
\end{pmatrix}.
\]

- Finally the SFIM \( \bar{I}(\theta_0|g_0) \) can be obtained as:

\[
\bar{I}(\theta_0|g_0) = E_0\{\bar{s}_0\bar{s}_0^T\}
\]

\[
= \begin{pmatrix}
C_0(\bar{s}_{\mu_0}) & 0 \\
0^T & C_0(\bar{s}_{\text{vecs}(\Sigma_0)})
\end{pmatrix},
\]

where \( C_0(h) \triangleq E_0\{hh^T\}, \forall h \in \mathcal{H}^q \).
Evaluation of the SCRB for the RES class

Through direct calculation of the expectation, we get:

$$C_0(\bar{s}_{\mu_0}) = \frac{4E\{Q\psi_0(Q)^2\}}{N} \Sigma_0^{-1},$$

and

$$C_0(\bar{s}_{\text{vecs}(\Sigma_0)}) = \frac{2E\{Q^2\psi_0(Q)^2\}}{N(N + 2)} \times$$

$$\times D_N^T \left( \Sigma_0^{-1} \otimes \Sigma_0^{-1} - \frac{1}{N} \text{vec}(\Sigma_0^{-1})\text{vec}(\Sigma_0^{-1})^T \right) D_N.$$

The block-diagonal structure of $\tilde{I}(\theta_0|g_0)$ implies that the estimates of vector $\mu_0$ and $\Sigma_0$ are asymptotically decoupled.

$\mu_0$ can be substituted with any consistent estimator without affecting the asymptotic performance of the scatter matrix estimator.
Evaluation of the SCRB for the RES class

Step D: Evaluation of the constrained $\text{SCRB}(\theta_0|g_0)$

- To avoid the scale-ambiguity problem, we need to put a constraint on $\Sigma_0$, i.e. $c(\Sigma_0) = 0$.

- Let $J_c(\Sigma_0)$ be the Jacobian matrix of the constraint, then there exists a matrix $U$ s.t. [31,32]:

$$J_c(\Sigma_0)U = 0, \quad U^T U = I.$$ 

- The constrained $\text{SCRB}(\theta_0|g_0)$ can be expressed as:

$$\text{CSCRB}(\theta_0|g_0) = \left( \begin{array}{cc} \frac{N}{4E\{Q_0(Q)^2\}} \Sigma_0 & 0 \\ \Sigma_0 & U \left(U^T C_0(\bar{s}_{\text{vecs}}(\Sigma_0))U \right)^{-1} U^T \end{array} \right).$$
Numerical results

- Let \( \{x_m\}_{m=1}^M \) be a set of \( M \) i.i.d. RES-distributed data, s.t.:
  \[
x_m \sim RES_N(x; \mu_0, \Sigma_0, g_0), \quad m = 1, \ldots, M.
\]

- Let us define \( \{\bar{x}_m\}_{m=1}^M \) as the set of \( M \) vectors such that:
  \[
  \bar{x}_m = x_m - \hat{\mu}, \quad m = 1, \ldots, M,
\]
  and \( \hat{\mu} \) is the sample mean estimator, i.e.
  \[
  \hat{\mu} \triangleq M^{-1} \sum_{m=1}^M x_m.
\]

- \( \hat{\mu} \) is a consistent and unbiased estimator.
Three “semiparametric” estimators (1/3)

- The efficiency w.r.t. the CSCRB of three estimators is investigated:
  - the constrained Sample Covariance matrix (CSCM),
  - the constrained Tyler’s estimator (C-Tyler),
  - the constrained Huber’s estimator (C-Hub).

- We impose a constraint on the trace: $\text{tr}(\Sigma_0) = N$.

- The CSCM is given by:

$$
\hat{\Sigma}_{SCM} \triangleq \frac{1}{M} \sum_{m=1}^{M} \bar{x}_m \bar{x}_m^T, \\
\hat{\Sigma}_{CSCM} \triangleq \frac{N}{\text{tr}(\hat{\Sigma}_{SCM})} \hat{\Sigma}_{SCM},
$$
The C-Tyler and the C-Hub are given by the convergence point of the following recursion:

\[
\begin{align*}
\mathbf{S}^{(k+1)}_T &= \frac{1}{M} \sum_{m=1}^{M} \varphi(t^{(k)}) \bar{x}_m \bar{x}_m^T, \\
\hat{\Sigma}^{(k+1)}_T &= NS^{(k+1)}_T / \text{tr}(S^{(k+1)}_T)
\end{align*}
\]

where \( t^{(k)} = \bar{x}_m^T (\hat{\Sigma}^{(k)}_T)^{-1} \bar{x}_m \) and the starting point is \( \hat{\Sigma}^{(0)}_T = I \).

The weight function \( \varphi(t) \) for Tyler’s estimator is [33,8]:

\[ \varphi_{\text{Tyler}}(t) = N/t, \]
The weight function for Huber’s estimator is given by \([24,34]\)

\[
\varphi_{Hub}(t) = \begin{cases} 
\frac{1}{b} & t \leq \delta^2 \\
\delta^2/(tb) & t > \delta^2 
\end{cases}
\]

and

\[
\delta = F_{\chi_N^2}(u), \quad 18 \\
b = F_{\chi_{N+2}^2}(\delta^2) + \delta^2(1 - F_{\chi_N^2}(\delta^2))/N \quad [8], [34].
\]

- \(u\) is a tuning parameter that controls the trade-off between robustness and efficiency.

- For \(u \rightarrow 1\) Huber’s estimator is equal to the SCM, while for \(u \rightarrow 0\) Huber’s estimator tends to Tyler’s estimator.

\(^{18}\) \(F_{\chi_N^2}(\cdot)\) indicates the distribution of a chi-squared random variable with \(N\) degrees of freedom.
Simulation setup

- Two different “true” distributions are considered:
  1. The t-distribution,
  2. The Generalized Gaussian (GG) distribution.

- Simulation parameters
  - $[\Sigma_0]_{i,j} = \rho^{|i-j|}$, $\rho = 0.8$ $i,j = 1,\ldots,N$. Moreover $N = 8$,
  - The data power is chosen to be $\sigma_X^2 = E_Q\{Q\}/N = 4$,
  - The data mean value is chosen to be $[\mu_0]_i = 1$, $i = 1,\ldots,N$,
  - The number of the available i.i.d. data vectors is $M = 3N = 24$,
  - The tuning parameter $u$ of Huber’s estimator $u = 0.5$.

- The MSE of the scatter matrix estimators is compared with:
  1. The CSCRBB(\theta_0|g_0) previously derived,
  2. The classical constrained CRB, i.e. CCRBB(\theta_0), evaluated under perfect knowledge of the density generator [35,36].
\[ \varepsilon_{\mu_0} \triangleq \| E\{(\hat{\mu} - \mu_0)(\hat{\mu} - \mu_0)^T\} \|_F, \quad \varepsilon_{\text{CSCRB},\mu_0} \triangleq \| [\text{CSCRB}(\theta_0|g_0)]_{\mu_0} \|_F. \]

- For the estimation of \( \mu_0 \), CSCRB coincides with CCRB.
- When the shape parameter \( \lambda \) goes to infinity, the \( t \)-distribution tends to a Gaussian one.
- Then, for \( \lambda \to \infty \), the sample mean tends to be efficient.
\[ \varepsilon_\alpha \triangleq \| E \{ \text{vecs}(\hat{\Sigma}_\alpha) - \text{vecs}(\Sigma_0)) (\text{vecs}(\hat{\Sigma}_\alpha) - \text{vecs}(\Sigma_0))^T \} \|_F, \]

\[ \varepsilon_{CSCRB,\Sigma_0} \triangleq \| [\text{CSCRB}(\theta_0|g_0)]\Sigma_0 \|_F, \quad \varepsilon_{CCRBF,\Sigma_0} \triangleq \| [\text{CCRBF}(\theta_0)]\Sigma_0 \|_F. \]

\[ \begin{align*}
\text{Shape parameter: } \lambda & \quad \text{MSE indices & Bounds} \\
\varepsilon_{C\text{SCM}} & \quad \varepsilon_{C\text{--Tyler}} \\
\varepsilon_{C\text{--Hub}(0.5)} \\
\varepsilon_{CCRBF,\Sigma_0} & \quad \varepsilon_{C\text{SCRB},\Sigma_0}
\end{align*} \]

- The CSCM tends to be efficient w.r.t. the CSCRB as \( \lambda \to \infty. \)
- Both C-Tyler’s and C-Huber’s estimators are not efficient with respect to the CSCRB.
\[
\varepsilon_{\mu_0} \triangleq \|E\{(\hat{\mu} - \mu_0)(\hat{\mu} - \mu_0)^T\}\|_F, \quad \varepsilon_{CSCRB,\mu_0} \triangleq \|[CSCRB(\theta_0|g_0)]\mu_0\|_F.
\]

- When \(s = 1\), the GG distribution is exactly Gaussian one.
- Hence, for \(s = 1\), the sample mean is an efficient estimator.
$\varepsilon_\alpha \triangleq ||E\{(\text{vecs}(\hat{\Sigma}_\alpha) - \text{vecs}(\Sigma_0))(\text{vecs}(\hat{\Sigma}_\alpha) - \text{vecs}(\Sigma_0))^T\}||_F,$

$\varepsilon_{\text{CSCR}}_0 \triangleq ||[\text{CSCR}(\theta_0|g_0)]\Sigma_0||_F, \quad \varepsilon_{\text{CCR}}_0 \triangleq ||[\text{CCR}(\theta_0)]\Sigma_0||_F.$

The lack of knowledge of the particular density generator has an higher impact when the tails of the true distribution become lighter [37].
The SCRB for the CES class

The derivation of:\(^{19}\)

- SCRB for the estimation of the mean vector and of the scatter matrix in CES distributed random vectors,
- The Semiparametric Slepian-Bangs formula,
- The Semiparametric Stochastic CRB (SSCRB),

can be found in [38]:


The application of these theoretical results to Direction of Arrival (DOA) estimation problems is discussed in [39]:


\(^{19}\) Additional details are given in the backup slides.
Conclusions

- We provided a fresh look to the Semiparametric Cramér-Rao Bound (SCRB) by showing its relations with the classical (parametric) CRB [7].

- The link between parametric and semiparametric framework is given by the Hilbert-space geometry underling any inference problem.

- The application of the SCRB to the scatter matrix estimation in RES and CES distributed data has been discussed.

- Future works will explore possible applications of the semiparametric inference to well-known signal processing problems, in particular the semiparametric detection.
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References


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Backup slides
\( \sigma \)-algebras and measures

- Let \( \mathcal{X} \) be some set and let \( 2^\mathcal{X} \) represent its power set. Then a subset \( \mathcal{F} \subseteq 2^\mathcal{X} \) is called a \( \sigma \)-algebra if (see e.g. [26, Ch. 2]):
  1. \( \mathcal{X} \in \mathcal{F} \),
  2. If \( A \in \mathcal{X} \) is in \( \mathcal{F} \), then so is its complement, \( \mathcal{X} \setminus A \),
  3. If \( \{ A_i \}_{i \in \mathbb{N}} \in \mathcal{F} \), then so \( \bigcup_{i=1}^\infty A_i \in \mathcal{F} \).

- A function \( \mu : \mathcal{F} \rightarrow [0, \infty) \) is called a measure if:
  1. \( \mu(\emptyset) = 0 \) (Null empty set),
  2. For all countable collections \( \{ A_i \}_{i=1}^\infty \) of pairwise disjoint sets in \( \mathcal{F} \), \( \mu \left( \bigcup_{i=1}^\infty A_i \right) = \sum_{i=1}^\infty \mu(A_i) \) (Countable additivity).

- The couple \( (\mathcal{X}, \mathcal{F}) \) is a measurable space, while the triplet \( (\mathcal{X}, \mathcal{F}, \mu) \) is a measure space.
Probability spaces and random variables

- A probability space is a measure space \((\Omega, \mathcal{D}, P)\) where:
  1. \(\Omega\) is the sample space that represents the set of all possible outcomes of a random experiment,
  2. \(\mathcal{D}\) is the \(\sigma\)-algebra on \(\Omega\),
  3. \(P\) is a probability measure, that is a measure \(P : \mathcal{D} \rightarrow [0, 1]\) satisfying \(P(\Omega) = 1\).

- Let \((\Omega, \mathcal{D}, P)\) be a probability space and \((\mathcal{X}, \mathcal{F})\) a measurable space.

A random variable (r.v.) \(X\) is a measurable function \(X : \Omega \rightarrow \mathcal{X}\), that is for every subset \(A \in \mathcal{F}\), its preimage

\[
X^{-1}(A) \triangleq \{\omega \in \Omega | X(\omega) \in A\},
\]

is an element of the \(\sigma\)-algebra \(\mathcal{D}\), i.e. \(X^{-1}(B) \in \mathcal{D}\).
Distribution and density functions

- A r.v. allows us to “transport” the probability structure, defined in the abstract space \((\Omega, \mathcal{D}, P)\), in \((\mathcal{X}, \mathcal{F})\).

- Specifically, a new probability measure can be defined on \((\mathcal{X}, \mathcal{F})\) as follows:

  \[
P_X(A) \triangleq P(\{\omega \in \Omega | X(\omega) \in A\}) = P(X^{-1}(A)), \quad A \in \mathcal{F}.
  \]

- Consequently, the triplet \((\mathcal{X}, \mathcal{F}, P_X)\) is a probability space.

- **Example:** If \(\mathcal{X} \equiv \mathbb{R}\) and \(\mathcal{F}\) is the Borel \(\sigma\)-algebra on \(\mathbb{R}\), then \(P_X\) is the distribution of \(X\) [26, Ch. 11].

- The **density** \(p_X\) of \(X\) is a measurable function satisfying:

  \[
P_X((-\infty, x]) = \int_{-\infty}^{x} p_X(a) \, da, \quad \forall x \in \mathbb{R}.
  \]
Sub-$\sigma$-algebra generated by a transformation

- Let $(\mathcal{X}, \mathcal{F}, P_\mathcal{X})$ be a probability space as previously defined.

- Let $T : (\mathcal{X}, \mathcal{F}) \rightarrow (\mathcal{Y}, \mathcal{L})$ a measurable transformation on $\mathcal{X}$.

- The preimage of $T$, i.e.:

  $$G(T) \triangleq \{ G \in \mathcal{F} | G = T^{-1}(A), A \in \mathcal{L} \}$$

  may be a coarser subset of $\mathcal{F}$!

- It can be shown that $G(T)$ is a $\sigma$-algebra [26, Theo. 8.1] and, clearly, $G(T) \subseteq \mathcal{F}$.

- $G(T)$ is then indicated as the sub-$\sigma$-algebra generated by the transformation $T$ [26, Def. 23.3].
Proof: Finite-dimensionality of the linear span

**Theorem**

Let \( \mathbf{u} = (u_1, \cdots, u_k)^T \) be a column vector of \( k \) arbitrary elements of an infinite-dimensional Hilbert space \( \mathcal{F} \). The linear span of \( \mathbf{u} \), defined as:

\[
\mathcal{V} \triangleq \{ \mathbf{v} | \mathbf{v} = \mathbf{A}\mathbf{u}, \mathbf{A} \text{ is any matrix in } \mathbb{R}^{q \times k} \},
\]

is a finite-dimensional subspace of \( \mathcal{F}^q \). Moreover, if \( u_1, \cdots, u_k \) are linearly independent in \( \mathcal{F} \), then \( \dim(\mathcal{V}) = kq \).

**Proof**

- Assume that the entries of \( \mathbf{u} \) are linearly independent.
- The dimension of a (finite-dimensional) space is equal to the minimum number of linearly independent vectors required to span it.
Proof: Finite-dimensionality of the linear span

Then if $\mathcal{V}$ has dimension $qk$, there must exist $qk$ linearly independent $q$-dimensional vectors such that $\mathcal{V} = \text{span}\{v_{11}, \ldots, v_{1k}, v_{q1}, \ldots, v_{qk}\}$.

Each vector $v_{ij}$, $i = 1, \ldots, q; j = 1, \ldots, k$ can be constructed by putting all except the $i$-th entry equal to 0 and the $i$-th entry equal to $u_j \in \mathcal{F}$ for $j = 1, \ldots, k$, i.e:

$\begin{bmatrix}
v_{11} & \cdots & v_{1k} & v_{21} & \cdots & v_{2k} \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}$

By visual inspection, it is immediate to verify that they are linearly independent and this concludes the proof.
A CES (zero-mean) random vector $\mathbf{x} \in \mathbb{C}^N$ admits a pdf [8]:

$$p_{\mathbf{X}}(\mathbf{x}; \Sigma) = c_{N,g} \left| \Sigma \right|^{-1} g(\mathbf{x}^H \Sigma^{-1} \mathbf{x}) \triangleq CES_N(\mathbf{x}; \Sigma, g),$$

where $G \ni g : \mathbb{R}^+_0 \to \mathbb{R}^+$ is the density generator and

$$G \triangleq \{ g : \mathbb{R}^+_0 \to \mathbb{R}^+ \mid \int_0^\infty t^{N-1} g(t) dt < \infty \}$$

The set of all CES pdfs is a semiparametric model of the form:

$$\mathcal{P}_{\Sigma,g} \triangleq \{ p_{\mathbf{X}} \mid p_{\mathbf{X}}(\mathbf{x} \mid \Sigma, g), \Sigma \in \mathcal{M}_N, g \in G \}.$$  

How can we build a parametric submodel of $\mathcal{P}_{\Sigma,g}$?
The set of all the density generator $\mathcal{G}$ is a convex set!

**Proof**

For every $g_0, g_1 \in \mathcal{G}$ and for every $\eta \in [0, 1]$, we have that:

1. $\eta g_1(t) + (1 - \eta) g_0(t)$ is a function of $t \triangleq x^H \Sigma^{-1} x$,

2. By linearity, $\int_0^{\infty} t^{N-1} [\eta g_1(t) + (1 - \eta) g_0(t)] dt < \infty$,

then $\eta g_1 + (1 - \eta) g_0 \in \mathcal{G}$ and consequently $\mathcal{G}$ is a convex set.

Then it is immediate to verify that:

$$CES_N(x; \Sigma, g_0) = CES_N(x; \Sigma, \eta g_1 + (1 - \eta) g_0)$$

$$= \eta CES_N(x; \Sigma, g_1) + (1 - \eta) CES_N(x; \Sigma, g_0).$$

$\mathcal{P}_{\Sigma, g}$ is a convex set as well!
Let us define a smooth parametric map as:

\[ \nu_i : [0, 1] \rightarrow \mathcal{G} \]

\[ \eta \mapsto \nu_i(t, \eta) \triangleq \eta g_i(t) + (1 - \eta) g_0(t), \]

where \( g_i \) is a generic density generator while \( g_0 \) is the true one.

The relevant \( i \)-th parametric submodel is then given by:

\[ \mathcal{P}_{\Sigma, \nu_{\eta_i}} = \{ p_X | p_X(x | \Sigma, \eta g_i + (1 - \eta) g_0), \Sigma \in \mathcal{M}_N, \eta \in [0, 1] \} \].

It is immediate to verify that this submodel satisfies the conditions C0, C1 and C2 given in slide 32.

In particular, Condition C2 is verified by choosing \( \eta = 0 \).
Let $p_X(x|\theta)$ be a parametric pdf with $\theta \in \Theta \subset \mathbb{R}^d$.

We indicate with $u_\theta(x)$ the following parametric map:

$$u_\theta : \Theta \to L_2$$

$$\theta \mapsto u_\theta(x) \triangleq \sqrt{p_X(x|\theta)},$$

$u_\theta$ is Hellinger (Fréchet) differentiable in $\theta_0$ if there exists a vector $\tilde{u}_{\theta_0} \equiv \tilde{u}_{\theta_0}(x)$ such that:

$$\|u_{\theta_0} + h - u_{\theta_0} - \tilde{u}_{\theta_0}^T h\| = o\left(\sum_i h_i^2\right), \quad h \to 0,$$

where $\|u_\theta\|^2 = \langle u_\theta, u_\theta \rangle = \int u_\theta^2(x) dx$.

$\tilde{u}_{\theta_0} \equiv \tilde{u}_{\theta_0}(x)$ is the Hellinger derivative of $u_\theta$ in $\theta_0$. 
Since $u_\theta(x) \triangleq \sqrt{p_X(x|\theta)}$, we have that:

$$\|u_\theta\|^2 = \langle u_\theta, u_\theta \rangle = \int p_X(x|\theta) dx = 1, \quad \forall \theta \in \Theta.$$ 

$u_\theta$ can be interpreted as a differentiable map between $\Theta$ and (a subset of) the surface $S(L_2)$ of the unit sphere in $L_2$.

Given a point on $S(L_2)$, say $u_{\theta_0}$, the tangent space $S \subseteq L_2$ of $S_0$ at $u_{\theta_0}$ is defined by the orthogonality condition:

$$\langle r, u_{\theta_0} \rangle = 0 \iff r \in S.$$ 

Note that the tangent space $S_0$ is a subset of $L_2$, while previously we defined it as a subset of $\mathcal{H}$.\(^{20}\)

\(^{20}\)Remember that $\mathcal{H} = \{ h : \mathcal{X} \rightarrow \mathbb{R} \mid E_X \{ h \} = 0, E_X \{|h|^2\} < \infty \}$. 
A geometrical intuition (2/4)
A geometrical intuition (3/4)

► Are the two definitions consistent?

► Let us define the (locally) one-to-one transformation:

\[ H_0 : S \rightarrow \mathcal{H} \]

\[ r \mapsto H_0(r) \triangleq \frac{2r}{u_{\theta_0}} = h. \]

► Then, we have:

\[ r \in S \Rightarrow \langle r, u_{\theta_0} \rangle = \int r(x)u_{\theta_0}(x)dx = 0 \]

\[ 2^{-1} \int h(x)u_{\theta_0}^2(x) = 2^{-1} \int h(x)p(x|\theta_0)dx = 0 \]

\[ \Rightarrow \mathbb{E}_x\{h\} = 0 \Rightarrow h \in \mathcal{H}. \]
The vice-versa is as follows:

\[ h \in \mathcal{H} \Rightarrow E_X \{ h \} = \int h(x)p(x|\theta_0)dx = 0 \]
\[ \Rightarrow 2 \int r(x)u_{\theta_0}^{-1}(x)p(x|\theta_0)dx = 2 \int r(x)u_{\theta_0}(x)dx = 0 \]
\[ \Rightarrow \langle r, u_{\theta_0} \rangle = 0 \Rightarrow r \in S. \]

Then the two definition are consistent [9, Sec. 3.1, Prep. 3]:

\[ \langle r, u_{\theta_0} \rangle = 0, \ \forall r \in S \iff E_X \{ h \} = 0, \ \forall h \in \mathcal{H}. \]
Recall that the score vector of $p_X(x|\theta)$ in $\theta_0$ is defined as:

$$s_{\theta_0} \triangleq \nabla_\theta \ln p_X(x|\theta_0).$$

If for all $\theta \in \Theta \subseteq \mathbb{R}^q$ [9, Sec. 2.1, Prep. 1]:

- $p_X(x|\theta)$ is continuously differentiable in $\theta$ for almost all $x$,

- $\left(\sum_i [s_{\theta_0}]_i^2\right)^{1/2} \in L_2(P_0),$

then [9, Sec. 2.1], we have that:

$$\hat{u}_{\theta_0} = \frac{1}{2} u_{\theta_0} s_{\theta_0}, \quad \hat{u}_{\theta_0} \in S^q, \; s_{\theta_0} \in \mathcal{H}^q.$$
The Semiparametric CRB (SCRB)

\[ \bar{s}_{0,i} \triangleq s_{\theta_0} - \Pi(s_{\theta_0}|T_{\eta_0,i}) \]

\[ \bar{s}_0 \triangleq s_{\theta_0} - \Pi(s_{\theta_0}|T_{g_0}) \]

\[ T_{\eta_0,i} \subseteq T_{g_0}, \forall i \in \mathcal{I} \]

\[ \|\bar{s}_{0,i}\| \geq \|\bar{s}_0\|, \forall i \in \mathcal{I} \]

\[ \Rightarrow E_0\{\bar{s}_{0,i}\bar{s}_{0,i}^T\} \geq E_0\{\bar{s}_0\bar{s}_0^T\} \triangleq \bar{I}(\theta_0|g_0) \]
The Least Favourable Submodel (1/2)

The Least Favourable Submodel (LFS) (if it exists) is the $i$-th parametric submodel of $\mathcal{P}_{\theta,g}$ s.t.:

$$\sup_{\{\mathcal{P}_{\theta,\nu_i}\}} \left[ E_0\{\bar{s}_0,i \bar{s}^T_0,i\}\right]^{-1} = \max_{\{\mathcal{P}_{\theta,\nu_i}\}} \left[ E_0\{s_0,i s^T_0,i\}\right]^{-1} = \bar{I}(\theta_0 | \nu_i)^{-1},$$

Let us define as Least Favourable Direction (LFD) the score vector [9, Sec. 3.1], [11, Sec. 2.2]:

$$s_{\eta_0,i}(x) = \nabla \eta \ln p_X(x | \gamma_0, \nu_i(x, \eta_0)),$$

Then, as shown previously, for the parametric case:

$$\Pi(s_{\theta_0} | \mathcal{T}_{n_0,i}) = E_0\{s_{\theta_0} s^T_{n_0,i}\} C_0(s_{n_0,i})^{-1} s_{n_0,i}.$$
The existence of a LFS depends on the “level of richness” of the set of the parametric submodels $\{P_{\theta,\nu_i}\}_{i \in I}$.

Unfortunately, the existence of a LFS needs to be verified on a case-by-case basis.

Moreover, if it exists, figuring out which such LFS is, is not an easy task (see [11] for some hints on this).

We refer to [9] for an exhaustive list of semiparametric models that admits a LFS expressible in “closed-form”.
Let $h \equiv h(X)$ be a function of the random variable (r.v.) $X$.

We defined the conditional expectation as $E\{h(X)|Y\}$ as the unique function of the r.v. $Y$ such that:

$$E\{[h(X) - E\{h(X)|Y\}]Y\} = 0.$$ 

The explicit “operative definition” of $E\{h(X)|Y\}$ is:

$$E\{h(X)|Y\} \triangleq \int_{x} h(x)p_{X|Y}(x|y)dx$$

$$= \int_{x} h(x)\frac{p_{X,Y}(x,y)}{p_{Y}(y)}dx,$$

where $p_{X,Y}$ is the joint pdf of $X$ and $Y$, $p_{X|Y}$ is the conditional pdf of $X$ given $Y$ and $p_{Y}$ is the pdf of $Y$. 

Conditional expectation: a remark (1/2)
Are the two definitions consistent?

\[ E\{[h(X) - E\{h(X)|Y]\}Y\} = 0 \Rightarrow \]

\[ \int_{X,Y} [h(X) - E\{h(X)|Y = y\}]p_{X,Y}(x, y)dxdy = 0 \]

\[ \int_{X,Y} h(x)p_{X,Y}(x, y)dxdy \]

\[ = \int_{X,Y} E\{h(X)|Y = y\}p_{X,Y}(x, y)dxdy \]

\[ = \int_{Y} E\{h(X)|Y = y\}p_{Y}(y)dy \]

\[ = \int_{Y} \left[ \int_{X} h(x)\frac{p_{X,Y}(x, y)}{p_{Y}(y)}dx \right] p_{Y}(y)dy \]

\[ = \int_{X,Y} h(x)p_{X,Y}(x, y)dxdy. \]
From RES to CES distributions (1/3)

Definition ([40], [28], [8] and [41, Ch. 4])

Let \( x_R \in \mathbb{R}^N \) and \( x_I \in \mathbb{R}^N \) be two real random vectors.

\( z \triangleq x_R + jx_I \in \mathbb{C}^N \) is said to be CES-distributed with mean vector \( \mu \) and scatter matrix \( \Sigma \):

\[
\mu = \mu_R + j\mu_I \in \mathbb{C}^N \quad \Sigma = C_1 + jC_2 \in \mathbb{C}^{N \times N},
\]

iff \( \tilde{x} \triangleq (x_R^T, x_I^T)^T \in \mathbb{R}^{2N} \) is RES-distributed with mean vector \( \tilde{\mu} = (\mu_R^T, \mu_I^T)^T \) and scatter matrix \( \tilde{\Sigma} \) satisfying:

\[
\tilde{\Sigma} = \frac{1}{2} \begin{pmatrix} C_1 & -C_2 \\ C_2 & C_1 \end{pmatrix},
\]

where \( C_1 \) is symmetric and \( C_2 \) is skew-symmetric.
From RES to CES distributions (2/3)

Let \( \tilde{x} \sim RES_{2N}(\tilde{x}; \tilde{\mu}, \tilde{\Sigma}, g) \) be a RES-distributed random vector.

When the scatter matrix \( \tilde{\Sigma} \) has full rank, we have that:

\[
RES_{2N}(\tilde{x}; \tilde{\mu}, \tilde{\Sigma}, g) \triangleq p_{\tilde{x}}(\tilde{x}; \tilde{\mu}, \tilde{\Sigma}, g) \\
= 2^{-(2N)/2} |\tilde{\Sigma}|^{-1/2} g \left( (\tilde{x} - \tilde{\mu})^T \tilde{\Sigma}^{-1} (\tilde{x} - \tilde{\mu})^T \right) \\
= |\Sigma|^{-1} g \left( 2(z - \mu)^H \Sigma^{-1} (z - \mu) \right) \\
= p_{Z}(z; \mu, \Sigma, h) \triangleq CES_{N}(z; \mu, \Sigma, h),
\]

where \( h(t) \triangleq g(2t) \).

The functional form of the density generator remains unchanged except for the scaling factor 2 of its argument.
There exists a one-to-one mapping between a subset of the RES distributions and the (circular) CES distributions.

The semiparametric theory already developed for the RES class holds true for the CES class as well.

In particular, CES distributions are a **semiparametric group model** generated by the set of Complex Spherically Symmetric (CSS) distributions [28, Sec. 3.5] through the action of:

\[
\alpha(\mu, \Sigma) : \mathbb{C}^N \rightarrow \mathbb{C}^N, \ \forall \mu, \Sigma
\]

\[
\text{CSS}(g) \sim z \mapsto \alpha(\mu, \Sigma)(z) = \mu + \Sigma^{1/2}z.
\]
The SCRB for the CES class

- The steps to derive the SCRB for the CES class follow exactly the ones already discussed for the RES one.

- **Difference**: the mean vector $\mu$ and the scatter matrix $\Sigma$ are complex quantities!

- The Wirtinger or $\mathbb{CR}$-calculus has to be used to evaluate the derivatives [42,43,44,45,46,47,48,49].

- All the details can be found in [38].
Slepian-Bangs (SB) formula

- Introduced by Slepian and Bangs in [50] and [51], the SB formula has been extensively used for many years in array processing.

- The “classic” SB formula is a compact expression of the Fisher Information Matrix (FIM) for parameter estimation under a Gaussian data model [13, Appendix 3C].

- Specifically:
  - $\theta \in \Theta \subseteq \mathbb{R}^d$: deterministic parameter vector,
  - $z \sim CN(\mu(\theta), \Sigma(\theta))$: complex Gaussian random vector.

- Then the SB formula provides us with a closed-form expression of the FIM for the estimation of $\theta \in \Theta$. 
Semiparametric Slepian-Bangs (SSB) formula

- Generalizations to:
  1. Non-circular complex Gaussian distributions [52],
  2. CES distributions [36],
  3. Non-circular CES distributions [53],
  4. Model misspecification under Gaussianity assumption [1],
  5. Model misspecification under CES assumption [54],
  6. Semiparametric model under CES assumption [38].

- Let $\mathbb{C}^N \ni z \sim CES_N(\mu(\theta), \Sigma(\theta), h)$ be a CES-distributed random vector parameterized by $\theta \in \Theta \subseteq \mathbb{R}^d$.

- The semiparametric SB (SSB) formula in [38] provides the efficient FIM for the estimation of $\theta$ in the presence of an unknown, nuisance density generator $h \in \mathcal{G}$. 
Assume to have an array of $N$ sensors and $K$ narrowband sources impinging on the array from $\{\nu_1, \ldots, \nu_K\}$ directions.

Data snapshots $z_m \sim CES_N(z; 0, \Sigma(\nu, \Gamma, \sigma^2), h_0), \forall m$ whose density generator $h_0 \in \mathcal{G}$ is unknown and [55]:

$$\Sigma \equiv \Sigma(\nu, \Gamma, \sigma^2) = A(\nu)\Gamma A(\nu)^H + \sigma^2 I_N.$$

The SSCRB$(\nu_0|\zeta_0, \sigma_0^2, h_0)$ [38,39] generalizes the classical, Gaussian-based, SCRB [56,57] since:

1. The Gaussianity assumption is replaced by the more general CES assumption,
2. The additional infinite-dimensional \textit{nuisace} parameter $h_0$ is taken into account.